The Homotopy Perturbation Method for Solving the Modified Korteweg-de Vries Equation

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Z. Naturforsch. **63a,** 621 – 626 (2008); received March 3, 2008

The homotopy perturbation method (HPM) is employed successfully for solving the modified Korteweg-de Vries equation. In this method, the solution is calculated in the form of a convergent series with an easily computable component. This approach does not need linearization, weak nonlinearity assumptions or perturbation theory. The results show applicability, accuracy and efficiency of the HPM in solving nonlinear differential equations. It is predicted that the HPM can be widely applied in science and engineering problems.

Key words: Homotopy Perturbation Method; Nonlinear Phenomena; Modified KdV Equation.

1. Introduction

In the last past decades, directly seeking for exact solutions of nonlinear partial differential equations has become one of the central themes of perpetual interest in mathematical physics. Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Nonlinear equations are widely used to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Nonlinear equations also cover the following cases: surface waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystal. The wide applicability of these equations is the main reason why they have attracted that much attention by many mathematicians. However, they are usually very difficult to solve, either numerically or theoretically.

During the past decades, both mathematicians and physicists have devoted considerable effort to the study of exact and numerical solutions of nonlinear ordinary or partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented, for instance the Bäcklund transformation method [1], Darboux transformation method [2], Adomian's decomposition method [3 – 6], exp-function method [7] and variational iteration method (VIM) [8 – 16]. Many authors have applied the homotopy perturbation method (HPM) to various types of Korteweg-de Vries (KdV) equations [17 – 20].

The aim of the present paper is to extend the HPM to derive the numerical and exact solutions of the modified KdV equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1a}$$

with the initial condition

$$u(x,0) = f(x). \tag{1b}$$

In the HPM introduced by He [21 – 27] the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a "small parameter". Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [17 – 20, 28 – 34]. We extend the method to solve the modified KdV equation.

2. Basic Ideas of the HPM

To illustrate the basic idea of He's HPM, consider the general nonlinear differential equation

$$A(u) - f(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega \tag{2}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \boldsymbol{r} \in \Gamma, \tag{3}$$

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where A is a general differential operator, B a boundary operator, $f(\boldsymbol{r})$ a known analytic function, and Γ the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts, L and N, where L is linear and N is nonlinear. Therefore (2) can be written as

$$L(u)+N(u)-f(\boldsymbol{r})=0. \tag{4}$$

By using the homotopy technique, one can construct a homotopy $v(r,p):\Omega\times[0,1]\to R$ which satisfies

$$\begin{aligned}H(v,p)=(1-p)[L(v)-L(u_0)]\\ +p[A(v)-f(r)]=0,\quad p\in[0,1]\end{aligned} \tag{5a}$$

or

$$\begin{aligned}H(v,p)=L(v)-L(u_0)+pL(v_0)\\ +p[N(v)-f(r)]=0,\end{aligned} \tag{5b}$$

where $p\in[1,0]$ is an embedding parameter, and u_0 is the initial approximation of (2) which satisfies the boundary conditions. Clearly, we have

$$H(v,0)=L(v)-L(u_0)=0, \tag{6}$$

$$H(v,1)=A(v)-f(r)=0. \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopic in topology. If the embedding parameter p $(0\le p\le 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can naturally assume that the solution of (6) and (7) can be given as a power series in p, i. e.,

$$v=v_0+pv_1+p^2v_2+\dots, \tag{8}$$

and setting $p=1$ results in the approximate solution of (2) as

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\dots. \tag{9}$$

The convergence of series (9) has been proved by He [26]. It is worth to note that the major advantage of He's HPM is that the perturbation equation can be freely constructed in many ways (therefore is problem-dependent) by homotopy in topology, and the initial approximation can also be freely selected. Moreover, the construction of the homotopy for the perturbation problem plays a very important role for obtaining the desired accuracy [29]. The HPM will become a much more interesting method to solve nonlinear differential equations in science and engineering.

3. Applying the HPM to Find Exact Solitary Wave Solutions of the Modified KdV Equation

3.1. First Adaptation of the HPM

We consider the modified KdV equation

$$\frac{\partial u}{\partial t}+u^2u_x+u_{xxx}=0 \tag{10}$$

with the initial condition

$$u(x,0)=\frac{4\sqrt{2}\sin^2(kx)}{3-2\sin^2(kx)}, \tag{11}$$

where k is an arbitrary constant.

To solve (10) and (11) by the HPM, we construct the homotopy

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}\right)=p\left(-u^2\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3}-\frac{\partial u_0}{\partial t}\right) \tag{12}$$

and assume the solution of (12) in the form

$$u=u_0+pu_1+p^2u_2+p^3u_3+\dots. \tag{13}$$

Substituting (13) into (12) and collecting terms of the same power of p gives

$$p^0:\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, \tag{14}$$

$$p^1:\frac{\partial u_1}{\partial t}=-{u_0}^2\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}-\frac{\partial u_0}{\partial t}, \tag{15}$$

$$p^2:\frac{\partial u_2}{\partial t}=-{u_0}^2\frac{\partial u_1}{\partial x}-2u_0u_1\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_1}{\partial x^3}, \tag{16}$$

$$\begin{aligned}p^3:\frac{\partial u_3}{\partial t}=-{u_0}^2\frac{\partial u_2}{\partial x}-2u_0u_1\frac{\partial u_1}{\partial x}\\ -2u_0u_2\frac{\partial u_0}{\partial x}-{u_1}^2\frac{\partial u_0}{\partial x}.\end{aligned} \tag{17}$$

The given initial value admits the use of

$$u_0(x,t)=\frac{4\sqrt{2}\sin^2(kx)}{3-2\sin^2(kx)}. \tag{18}$$

Thus, by solving the equations above, we obtain $u_1,u_2,\dots$, e. g.

$$u_1(x,t) = -\frac{96\sqrt{2}k^4\cos(kx)\sin(kx)}{1+4\cos^2(kx)+4\cos^4(kx)}t, \tag{19}$$

$$u_2(x,t) = -192\frac{\sqrt{2}k^7(4\cos^4(kx)-8\cos^2(kx)+1)}{1+6\cos^2(kx)+12\cos^4(kx)+8\cos^6(kx)}t^2, \tag{20}$$

$$u_3(x,t) = 1024\frac{\sqrt{2}k^{10}(4\cos^4(kx)-20\cos^2(kx)+7)\cos(kx)\sin(kx)}{1+8\cos^2(kx)+24\cos^4(kx)+32\cos^6(kx)+16\cos^8(kx)}t^3, \tag{21}$$

and so on; in the same manner the rest of components is obtained by MAPLE Package.

The approximate solutions of the problem can be readily obtained by

$$\begin{aligned}
&u0 = u_0 && \text{with zero-order approximation,}\\
&u1 = u_0+u_1 && \text{with first-order approximation,}\\
&u2 = u_0+u_1+u_2 && \text{with second-order approximation,}\\
&u3 = u_0+u_1+u_2+u_3 && \text{with third-order approximation,}\\
&\vdots
\end{aligned}$$

Consequently, we have the solution of (10) in the series form

$$\begin{aligned}
u(x,t) = &-\frac{4\sqrt{2}\sin^2(kx)}{3-2\sin^2(kz)} - \frac{96\sqrt{2}k^4\cos(kx)\sin(kx)}{1+4\cos^2(kx)+4\cos^4(kx)}t - 192\frac{\sqrt{2}k^7(4\cos^4(kx)-8\cos^2(kx)+1)}{1+6\cos^2(kx)+12\cos^4(kx)+8\cos^6(kx)}t^2\\
&+1024\frac{\sqrt{2}k^{10}(4\cos^4(kx)-20\cos^2(kx)+7)\cos(kx)\sin(kx)}{1+8\cos^2(kx)+24\cos^4(kx)+32\cos^6(kx)+16\cos^8(kx)}t^3+\dots.
\end{aligned} \tag{22}$$

The solution $u(x,t)$ in the closed form is

$$u(x,t) = \left\{\begin{array}{ll} \dfrac{4\sqrt{2}k\sin^2(kx-4k^3t)}{3-2\sin^2(kx-4k^3t)}, & 0 \le kx-4k^3t \le \pi, \\ 0, & \text{otherwise} \end{array}\right\}, \tag{23}$$

which is the exact solution of the problem.

The behaviour of the solutions obtained by the HPM and the approximate solution $u3$ (third-order approximation) are shown for different values of time in Figures 1 – 3.

3.2. Second Adaptation of the HPM

Again we consider the modified KdV equation

$$\frac{\partial u}{\partial t} + u^2 u_x + u_{xxx} = 0 \tag{24}$$

with the initial condition

$$u(x,t) = \frac{4\sqrt{2}\sin^2(kx)}{3-2\sin^2(kx)}, \tag{25}$$

where k is an arbitrary constant.

To solve (24) and (25) by the HPM, we construct the homotopy

$$\left(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}\right) = p\left(-u^2\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} - \frac{\partial u_0}{\partial t}\right) \tag{26}$$

and assume the solution of (26) in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots. \tag{27}$$

Substituting (27) into (26) and collecting terms of the same power of p gives

$$p^0: \quad \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \tag{28}$$

$$p^1: \quad \frac{\partial u_1}{\partial t} = -{u_0}^2\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_0}{\partial x^3} - \frac{\partial u_0}{\partial t}. \tag{29}$$

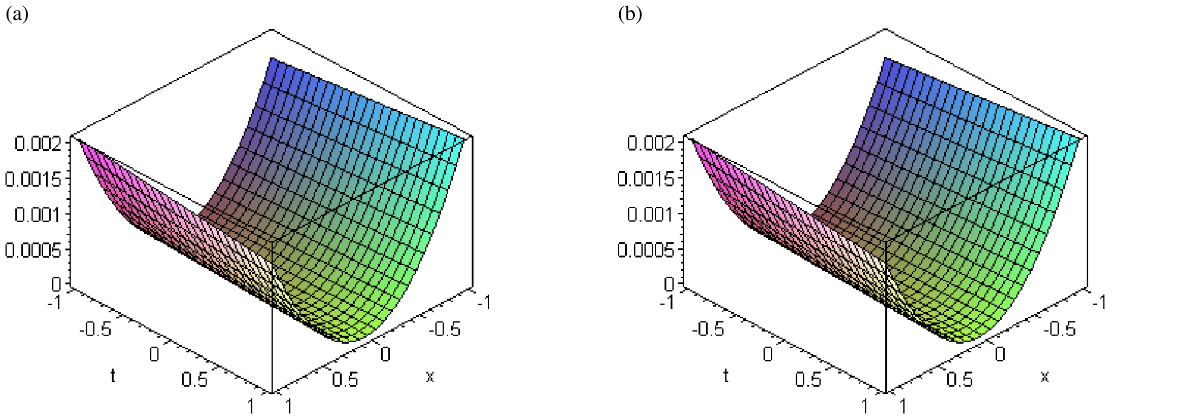

Fig. 1. (a) The numerical results for $u(x,t)$ by means of the homotopy perturbation method: $u3$ (third-order approximation). (b) The exact solution for $u(x,t)$ with $k = 0.1$ for different values of time ($-1 \le t \le 1$).

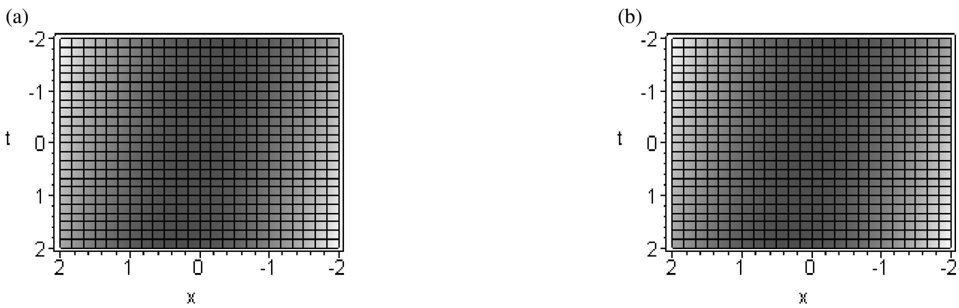

Fig. 2. (a) The density plot of the numerical results for $u(x,t)$ by means of the homotopy perturbation method: $u3$ (third-order approximation). (b) The density plot of the exact solution for $u(x,t)$ with $k = 0.2$ for different values of time ($-2 \le t \le 2$).

We change the initial function u_0; the given initial value admits the use of

$$u_0(x,t) = \frac{4\sqrt{2}\sin^2(kx+ct)}{3-2\sin^2(kx+ct)}. \tag{30}$$

Substituting (30) into (29), we obtain

$$\frac{\partial u_1}{\partial t} = \frac{32\sqrt{2}\cos(kx+ct)\sin(kx+ct)k^3}{3-2\sin^2(kx+ct)} + \left(-\frac{8\sqrt{2}\sin(kx+ct)c}{3-2\sin^2(kx+ct)} - \frac{-32\sqrt{2}\sin(kx+ct)k^3}{3-2\sin^2(kx+ct)}\right)t + \text{noise terms}. \tag{31}$$

No secular term in u_1 requires that [21, 22]

$$-\frac{8\sqrt{2}\sin(kx+ct)c}{3-2\sin^2(kx+ct)} - \frac{32\sqrt{2}\sin(kx+ct)k^3}{3-2\sin^2(kx+ct)} = 0, \tag{32}$$

and

$$c = -4k^3 \tag{33}$$

is obtained.

If we stop at the zero-order approximation and substitute (33) into (30), we obtain

$$u_0(x,t) = \frac{4\sqrt{2}\sin^2(kx-4k^3t)}{3-2\sin^2(kx-4k^3t)}, \tag{34}$$

which is the exact solution of the problem.

4. Conclusion

We have presented a scheme used to obtain numerical and exact solutions of the modified KdV equation with initial condition using the HPM. The approx-

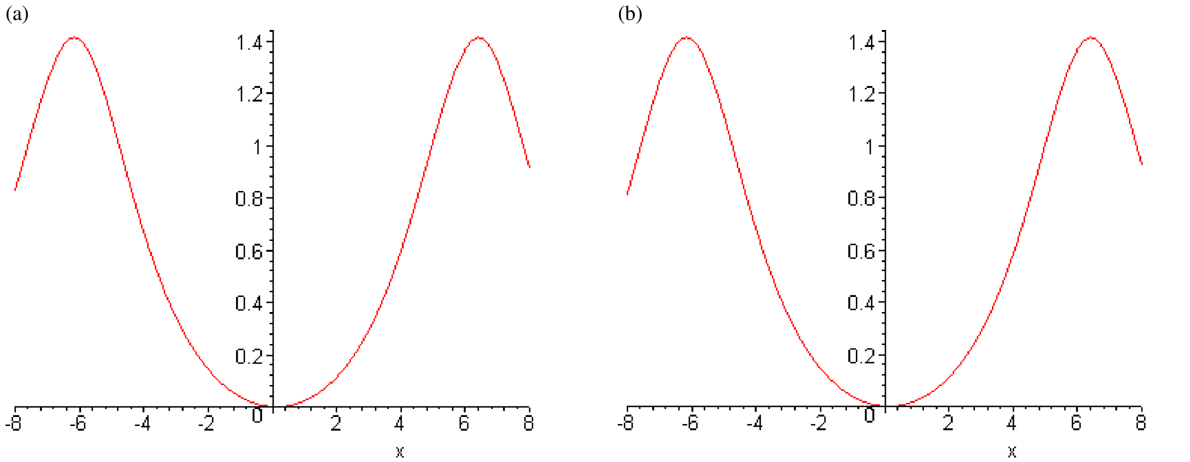

Fig. 3. (a) The numerical results for $u(x,t)$ by means of the homotopy perturbation method: $u3$ (third-order approximation). (b) The exact solution for $u(x,t)$ with $k = 0.25$ and $t = 0.5$ for different values of x ($-8 \le x \le 8$).

imate solutions were compared with the exact solutions in Figures 1 – 3. The method needs much less computational work compared with traditional methods. It is extremely simple, easy to use and very accurate for solving nonlinear equations. It was shown that the HPM is a very fast convergent, precise and cost efficient tool for solving nonlinear problems. The HPM will become a much more interesting method to solve nonlinear problems in science and engineering.

Acknowledgements

The author thanks TÜBİTAK (The Scientific and Technological Research Council of Turkey) for financial support.

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